Since $\sum_{n=3}^{\infty}\frac{\ln\ln n}{n^2}$ and $\sum_{n=3}^{\infty}\frac{1}{n^2\ln n}$ converge, so

$$1+\frac{1}{2}\cdot\sqrt{1+\frac{1}{2}}+\frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}}+\cdots+\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}=\ln(n)+O(1),$$

and we are done.

Editor's comment: **D. M. Bătinetu-Giurgiu, of the "Matei Basarab" National College in Bucharest, Romania and Neculai Stanciu, of George Emil Palade School in Buzău, Romania,** submitted two solutions to the problem. Their first solution was similar in approach to the second solution presented above, but in their second solution they generalized the problem as follows:

If $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are sequences of positive real numbers such that:

- $\{y_n\}_{n\geq 1}$ is increasing and unbounded,
- $\exists t\in\Re_+$ such that $\lim_{n\to\infty} n^t\left\{y_{n+1}-y_n\right\}=a\in\Re_+$,
- $\lim_{n\to\infty} n^t x_n=a$ exists $\in\Re_+$, and $z_n=\sum_{k=1}^{n}x_k$, then

  $\{y_n\}_{n\geq 1}\sim\{z_n\}_{n\geq 1}$. I.e., $\lim_{n\to\infty}\frac{z_n}{y_n}$.

Proof. By the Cesaro-Stolz theorem we have:

$$\lim_{n\to\infty}\frac{z_n}{y_n}=\lim_{n\to\infty}\frac{z_{n+1}-z_n}{y_{n+1}-y_n}=\lim_{n\to\infty}\frac{x_{n+1}}{y_{n+1}-y_n}=\lim_{n\to\infty}\frac{(n+1)^t x_{n+1}}{\left(\frac{n+1}{n}\right)^t n^t(y_{n+1}-y_n)}=\frac{a}{1\cdot a}=1.$$

Remark: If we take $y_n=\ln n$, $h_n=\sum_{k=1}^{n}\frac{1}{k}$, $x_n=\frac{1}{n}\sqrt[n]{h_n}$, and $z_n=\sum_{k=1}^{n}x_k$, then by the above we obtain $\{y_n\}_{n\geq 1}\sim\{z_n\}_{n\geq 1}$ which is problem 5257.

Also solved by Bruno Sagueiro Fanego, Viveiro, Spain; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy, and the proposer.

5258: *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Polytechnical University of Catalonia, Barcelona, Spain*

Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be real numbers such that $1+\sum_{k=1}^{n}\cos^2\alpha_k=n$. Prove that:

$$\sum_{1\leq i<j\leq n}\tan\alpha_i\tan\alpha_j\leq\frac{n}{2}.$$

Solution 1 by Arkady Alt, San Jose, CA

Let $x_i = \tan^2 \alpha_i, i = 1, 2, ..., n$ then $x_i \geq 0, i = 1, 2, ..., n, 1 + \sum\limits_{k=1}^{n} \cos^2 \alpha_k = n \iff$

$\sum\limits_{k=1}^{n} \dfrac{1}{1+x_i} = n - 1$ and, since $\sum\limits_{1\leq i<j\leq n}^{n} \tan\alpha_i \tan\alpha_j \leq \sum\limits_{1\leq i<j\leq n}^{n} |\tan\alpha_i| \, |\tan\alpha_j| = \sum\limits_{1\leq i<j\leq n}^{n} \sqrt{x_i x_j}$,

then it is sufficient to prove $\sum\limits_{1\leq i<j\leq n}^{n} \sqrt{x_i x_j} \leq \dfrac{n}{2}$.

Let $a_i = \dfrac{x_i}{1+x_i}, 1, 2, .., n$ then $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} \left(1 - \dfrac{1}{1+x_i}\right) = n - \sum\limits_{i=1}^{n} \dfrac{1}{1+x_i} = 1$ and,

since $x_i = \dfrac{a_i}{1-a_i}, 1, 2, ..., n$ our problem is:

Prove inequality $\sum\limits_{1\leq i<j\leq n}^{n} \sqrt{\dfrac{a_i a_j}{(1-a_i)(1-a_j)}} \leq \dfrac{n}{2},$ for $a_i \geq 0, i = 1, 2, ..., n$ such

that $\sum\limits_{k=1}^{n} a_i = 1$.

We have $\sum\limits_{1\leq i<j\leq n}^{n} \sqrt{\dfrac{a_i a_j}{(1-a_i)(1-a_j)}} \leq \sum\limits_{1\leq i<j\leq n}^{n} \dfrac{1}{2}\left(\dfrac{a_j}{1-a_i} + \dfrac{a_i}{1-a_j}\right) =$

$$\frac{1}{2}\left(\sum_{1\leq i<j\leq n}^{n} \frac{a_j}{1-a_i} + \sum_{1\leq i<j\leq n}^{n} \frac{a_i}{1-a_j}\right) = \frac{1}{2}\left(\sum_{i=1}^{n-1} {}_{j=i+1}^{n} \frac{a_j}{1-a_i} + \sum_{j=2}^{n} {}_{i=1}^{j} \frac{a_i}{1-a_j}\right) =$$

$$\frac{1}{2}\left(\sum_{i=1}^{n-1} \frac{1}{1-a_i} \sum_{j=i+1}^{n} a_j + \sum_{j=2}^{n} \frac{1}{1-a_j} \sum_{i=1}^{j-1} a_i\right) = \frac{1}{2} \cdot \frac{1}{1-a_1} \sum_{j=2}^{n} a_j + \frac{1}{2} \sum_{i=2}^{n-1} \frac{1}{1-a_i} \sum_{j=i+1}^{n} a_j +$$

$$\sum_{j=2}^{n-1} \tfrac{1}{1-a_j} \sum_{i=1}^{j-1} a_i + \frac{1}{1-a_n} \sum_{i=1}^{n-1} a_i = 1 + \frac{1}{2}\left(\sum_{i=2}^{n-1} \frac{1}{1-a_i} \sum_{j=i+1}^{n} a_j + {}_{j=2}^{n-1} \frac{1}{1-a_j} \sum_{i=1}^{j-1} a_i\right) =$$

$$1 + \frac{1}{2}\left(\sum_{i=2}^{n-1} \frac{1}{1-a_i} \sum_{j=i+1}^{n} a_j + \sum_{i=2}^{n-1} \frac{1}{1-ai} \sum_{j=1}^{i-1} a_i\right) = 1 + \frac{1}{2}\left(\sum_{i=2}^{n-1} \frac{1}{1-a_i}\left(\sum_{j=i+1}^{n} a_j + \sum_{j=1}^{i-1} a_i\right)\right) =$$

$$1 + \frac{1}{2}\sum_{i=2}^{n-1} \frac{1-a_i}{1-a_i} = 1 + \frac{n-2}{2} = \frac{n}{2}.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy

Proof: We first note that if

$\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0$, $\alpha_n = \pi/2$, the constraints of the problem are satisfied, but

$$\sum_{1\leq i<j\leq n} \tan\alpha_i \tan\alpha_j$$

is undefined; so we add the assumption $\alpha_i \neq \pi/2 + 2k\pi$, $k \in Z$,

$i = 1, \ldots, n$. Both $\cos^2 x$ and $\tan x$ are π–periodic so we can assume